October Math Gems

Problem of the week 32

§1 Problems

Problem 1.1. There are 45 rose and 18. These have to put into bouquets which contain both kinds. All bouquets should have the same number of flowers. Find the maximum number of bouquets and the number of flowers in them.

Problem 1.2. Use Euclid's division lemma to show that the cube of any positive integer is one of these three forms $(9k, 9k+1, 9k+8)$ note that Euclid division algorithm $= a = bq + r, 0 \leq r \leq b$
It indicate that for given positive integers a and b, there exist unique integers q and r satisfying the equation.

Problem 1.3. show that any positive even integer is one of these forms $(4q, 4q+2)$ where q is the whole number.

Problem 1.4. If the polynomial $x^2 - 2x + k$ is a factor of$x^4 - 6x^3 + 16x^2 - 26x + 10 - a$ find the value of a and k

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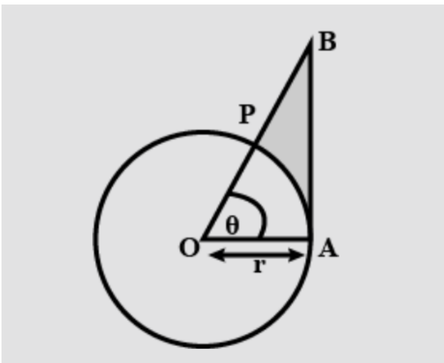
Problem 1.5. prove that the perimeter of shaded region is equal to

$$r[\tan\theta + \sec\theta + \frac{\pi\theta}{180} - 1]$$



Problem 1.6. Given that $3^x + 3^y = 10$,and $3^{xy} = 5$.Then, find

$$3^{x-y} + 3^{y-x}$$

Problem 1.7. Evaluate the value of expression in terms of x, $\cos(\tan^{-1}(\sin(\cot^{-1} x)))$

Problem 1.8. $\frac{1}{\log_2 m} + \frac{1}{\log_4 m} + \frac{1}{\log_8 m} \ldots\ldots + \frac{1}{\log_{2^n} m} = n(n+1)\alpha$
$\alpha =$

Problem 1.9. If α and β are the root of the equation $x^2 + ax + b = 0$, then $\frac{1}{\alpha^2} + \frac{1}{\beta^2} =$

Problem 1.10. Let P be a point inside triangle ABC such that the line AP is the bisector of the $\angle BAC$, BP is the bisector of $\angle ABC$ and CP is the bisector of $\angle ACB$. If $\angle APB = 110^o$, $\angle BPC = 120^o$ and $\angle CPA = 130^o$, find the angles of triangle ABC.

Problem 1.11. Let $ABCD$ be a quadrilateral inscribed in a circle such that $\angle ABC = 60°$. Prove that if $BC = CD$ then $AB = CD + DA$.

Problem 1.12. There is a positive real number x not equal to either $\frac{1}{20}$ or $\frac{1}{2}$ such that

$$\log_{20x}(22x) = \log_{2x}(202x).$$

The value $\log_{20x}(22x)$ can be written as $\log_{10}(\frac{m}{n})$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 1.13. Find the number of real roots of the equation

$$e^{4x} - e^{3x} - 4e^{2x} - e^x + 1$$

Problem 1.14. Find the sum of the following infinite series

$$1 + \frac{3}{\pi} + \frac{3}{\pi^2} + \frac{3}{\pi^3} + \ldots$$

Problem 1.15. There are real numbers a, b, c, and d such that -20 is a root of $x^3 + ax + b$ and -21 is a root of $x^3 + cx^2 + d$. These two polynomials share a complex root $m + \sqrt{n} \cdot i$, where m and n are positive integers and $i = \sqrt{-1}$. Find $m + n$.

Problem 1.16. Let a, b, c, and d be real numbers that satisfy the system of equations

$$\begin{aligned} a + b &= -3 \\ ab + bc + ca &= -4 \\ abc + bcd + cda + dab &= 14 \\ abcd &= 30. \end{aligned}$$

There exist relatively prime positive integers m and n such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Find $m + n$.

Problem 1.17. Two infinite geometric series have the same sum. The first term of the first series is 1, and the first term of the second series is 4. The fifth terms of the two series are equal. The sum of each series can be written as $m + \sqrt{n}$ where m and n are positive integers. Find $m + n$

Problem 1.18. By using heron's formula solve the following question
The side lengths of a scalene triangle are the roots of the polynomial

$$x^3 - 20x^2 + 131x - 281.3$$

Find the square of the area of the triangle.

Problem 1.19. Known that

$$\frac{\sin^4(x)}{2} + \frac{\cos^4(x)}{3} = \frac{1}{5}$$

Then, $\tan^2(x) =$

Problem 1.20. If $a\cos(A) + b\sin(A) = a\cos(B) + b\sin(B)$, show that

$$\cos^2(\frac{A+B}{2}) - \sin^2(\frac{A+B}{2}) = \frac{a^2 - b^2}{a^2 + b^2}$$
